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# A NEW NONCONFORMING RECOVERY ALGORITHM FOR TWO-SIDED BOUNDS OF EIGENVALUES

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**Abstract**

*In recent papers the authors present and develop an original method for obtaining two-sided bounds of the exact eigenvalues. This idea is based on the combination of nonconforming finite element methods giving lower bounds of eigenvalues and a postprocessing procedure using conforming finite element spaces. Here a new approach is proposed, applicable to second- and fourth-order eigenvalue problems. Namely, a conforming finite element method is used for eigenpairs approximation and then by means of nonconforming recovery interpolant a lower bound approximation of the exact eigenvalues is obtained when the mesh parameter $h > 0$ is sufficiently small.*

*Some appropriate combinations of finite elements (conforming and nonconforming) which fulfil the algorithm are presented and discussed. Numerical experiments illustrating the efficiency of the proposed method are also given.*

**Keywords:** finite element method; nonconforming/conforming elements; recovery interpolation; eigenvalue problem.

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## 1. INTRODUCTION

Eigenvalue problems arise in many physical and engineering applications. Because of the fact, that very few of them can be solved exactly, the two-sided bounds of eigenvalues is a very important tool in computation using finite element method (FEM) especially. Nowadays, the approaches for obtaining the eigenvalue approximations simultaneously from above and from below use different postprocessing procedures (see [1,2,3]).

In this paper, we define "nonconforming interpolations" of conforming eigenfunction approximations. This interpolation procedure gives lower bounds of eigenvalues. So, we just have to solve essentially one discrete eigenvalue problem.

The model eigenvalue problems are stated as follows:

$$\begin{aligned} -\Delta^m u &= \lambda u \quad \text{in } \Omega,\ m=1;2, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega \text{ if } m=2, \end{aligned} \tag{1}$$

where $\Omega$ is a bounded polygonal domain in $R^2$ with boundary $\partial\Omega$.

Let $H^s(\Omega)$ be the usual $s$-th order Sobolev space on $\Omega$ with a norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$ and $(\cdot,\cdot)$ denote the $L_2(\Omega)$ – inner product.

The variational elliptic eigenvalue problems associated with (1) are: find number $\lambda \in R$ and function $u \in H_0^m(\Omega)$, $m=1;2$ such that

$$\begin{aligned} a(u,v) &= \lambda\,(u,v), \quad \forall\, v \in V \equiv H_0^m(\Omega), \\ \|u\|_{0,\Omega} &= 1, \end{aligned} \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx\, dy \quad \forall\, u,v \in V,$$

or

$$a(u,v) = \int_\Omega \Delta u \;\Delta v \; dx\, dy \quad \forall\, u,v \in V,$$

for second- or fourth-order problem, respectively.

One sees that (2) has an eigenvalue sequence (see [4]):

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots,$$

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$lim_{k \to \infty} \lambda_k = \infty$.

The associated eigenfunctions $u_j$ can be orthonormalyzed in $L_2(\Omega)$ and they constitute a Hilbert basis for *V*.

## 2. FINITE ELEMENT METHOD

Let $\tau_h$ be a triangulation of $\Omega$ which satisfies the usual regularity conditions (see [5]), i.e. there exists a constant $\sigma > 0$ such that $h_K / \rho_K \le \sigma$, where $h_K$ is the diameter of the element $K \in \tau_h$ (rectangle or triangle) and $\rho_K$ being the diameter of the largest circle contained in *K*. Then we denote $h = max_{K \in \tau_h} h_K$.

Let $V_h$ be the finite element space consisting of piecewise polynomial functions of degree $\ge 2$ defined on $\tau_h$.

Then the approximation of the problem (2) is: find $\lambda_h \in R$ and function $u_h \in V_h$, $u_h \ne 0$ such that

$$a(u_h, v_h) = \lambda_h (u_h, v_h), \quad \forall v_h \in V_h . \tag{3}$$

We emphasize that $V_h \subset V$ (conforming FEM). Let us also introduce nonconforming finite element spaces $\widetilde{V}_h$ related to the partitions $\tau_h$. For this purpose we define mesh-dependent bilinear form

$$a_h(u,v) = \sum_{K \in \tau_h} a_K(u,v), \; u,v \in V,$$

where

$$a_K(u,v) = \int_K \nabla u \cdot \nabla v \, dx\, dy \quad \text{or} \quad a_K(u,v) = \int_K \Delta u \; \Delta v \, dx\, dy$$

for second- or fourth-order problem, respectively. In case of conforming FEM, obviously $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ coincide.

Our algorithm applies the so-called "nonconforming interpolation operator" $\widetilde{i}_h : V \to \widetilde{V}_h$ such, that

$$a_h(v - \widetilde{i}_h v, \widetilde{v}_h) = 0, \; \forall v \in V, \; \forall \widetilde{v}_h \in \widetilde{V}_h . \tag{4}$$

We specify a set of nonconforming finite elements with integral-type degrees of freedom:

(A) <u>For the second-order eigenvalue problems</u>:

- Linear triangular elements of Crouzeix-Raviart (*C-R*) [5];
- Bilinear rectangular elements of Rannacher-Turek ($Q_1^{rot}$) [6];
- The extensions of the elements above (*EC-R*, $EQ_1^{rot}$) [7,8].

(B) <u>For the fourth-order eigenvalue problems</u>:

- Triangular Morley element with degrees of freedom $v(a_j)$, $\frac{1}{\int_{l_j} dl} \int_{l_j} \partial_\nu v\, dl$, where $l_j$ is the edge of any $K \in \tau_h$ opposite of the vertex $a_j, j = 1,2,3$ [9];
- The rectangular version of Morley element with polynomial set $P_K = P_2 + span\{x^3 - 3xy^2, y^3 - 3yx^2\}$ [10], where $P_2$ denotes the set of all polynomials in two variables of degree less than or equal to 2.

**Lemma 1.** *If $\widetilde{V}_h$ is constructed by any elements described in (A) or (B) respectively, then the equality (4) is fulfilled.*

*Proof.* First, we consider triangular (*C-R* and Morley) finite elements for second- $(m = 1)$ and fourth-order $(m = 2)$ problem, respectively.

Then for any $K \in \tau_h$

$$\oint_{l_K} v\, dl = \oint_{l_K} \widetilde{i}_h v\, dl \qquad (C\text{-}R)$$

$$\oint_{l_K} \partial_\nu v\, dl = \oint_{l_K} \partial_\nu \left(\widetilde{i}_h v\right) dl \qquad \text{(Morley triangle)},$$

where $\partial_\nu$ is the outer normal derivative and $l_K$ are the edges of *K*.

Now, we adopt the following notations:
$\partial_\nu^0 v = v;\; \Delta^0 v = \nabla v$.

Then, when $v \in V$ and $\widetilde{v}_h \in \widetilde{V}$, we have

$$a_h(v - \widetilde{i}_h v, \widetilde{v}_h) = \sum_{K \in \tau_h} \int_K \Delta^{m-1}\left(v - \widetilde{i}_h v\right) \Delta^{m-1} \widetilde{v}_h \, dx\, dy, \; m = 1;2.$$

For any $K \in \tau_h$, $\widetilde{v}_{h|_K}$ is a polynomial from $P_m(K)$.

Using the Green formula and $\Delta^{m-1} \widetilde{v}_{h|_K} = const$, it follows

$$\int_K \Delta^{m-1}\left(v - \widetilde{i}_h v\right) \Delta^{m-1} \widetilde{v}_h \, dx\, dy = \oint_{\partial K} \Delta^{m-1} \widetilde{v}_h \, \partial_\nu^{m-1}\left(v - \widetilde{i}_h v\right) dl$$

$$= \Delta^{m-1} \widetilde{v}_{h|_K} \oint_{\partial K} \partial_\nu^{m-1}\left(v - \widetilde{i}_h v\right) dl = 0.$$

Into the last relation we use that

$$\oint_{\partial K} \partial_\nu^{m-1} v\, dl = \oint_{\partial K} \partial_\nu^{m-1} \widetilde{i}_h v\, dl,$$

for $m = 1$ (*C-R*) and $m = 2$ (Morley triangle), respectively.

The next considerations involve the nonconforming elements $Q_1^{rot}$, *EC-R* and $EQ_1^{rot}$, so that $m = 1$ and $\widetilde{V}_h$ consists of piecewise incompleted polynomials of degree two.

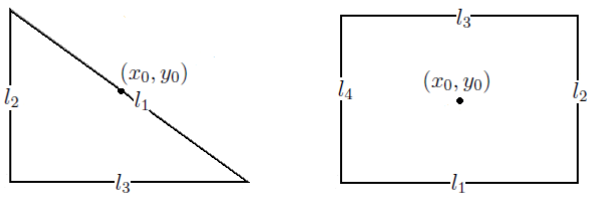
Let *T* be the reference element, so

$$T = \{(x, y) \in T : 0 \le x \le 1, \; 0 \le y \le 1 - x\}$$

if on *T* is defined the *EC-R* triangular element and

$T = \{(x, y) \in T : 0 \le x, y \le 1\}$

if $\Omega$ is discretized by means of $Q_1^{rot}$ or $EQ_1^{rot}$ -elements. These elements are depicted in Fig. 1 and also $x_0 = y_0 = \dfrac{1}{2}$ .




*Фиг. 1. The considered EC-R, $Q_1^{rot}$ and $EQ_1^{rot}$ -elements*

Since $T$ is an incomplete quadratic element, for $\tilde{v}_h \in \tilde{V}_h$ it can be written:

$$\tilde{v}_h(x,y) = \tilde{v}_h(x_0,y_0) + (x - x_0)\partial_x \tilde{v}_h(x_0,y_0)$$

$$+ (y - y_0)\partial_y \tilde{v}_h(x_0,y_0) + \frac{1}{2}(x - x_0)^2 \partial_{xx}\tilde{v}_h(x_0,y_0) \qquad (5)$$

$$+ \frac{1}{2}(y - y_0)^2 \partial_{yy}\tilde{v}_h(x_0,y_0).$$

In addition

$$\partial_x \tilde{v}_h(x,y) = \partial_x \tilde{v}_h(x_0,y_0) + (x - x_0)\partial_{xx}\tilde{v}_h;$$

$$\partial_y \tilde{v}_h(x,y) = \partial_y \tilde{v}_h(x_0,y_0) + (y - y_0)\partial_{yy}\tilde{v}_h.$$

Case 1. *EC-R*-element

In this case $\partial_{xx}\tilde{v}_h = \partial_{yy}\tilde{v}_h = const$ . Thus, we obtain:

$$\iint_T \partial_x\left(\tilde{i}_h v - v\right)\partial_x \tilde{v}_h \, dx\, dy = \iint_T \partial_x\left(\tilde{i}_h v - v\right)\partial_x \tilde{v}_h(x_0,y_0)\, dx\, dy$$

$$+ \iint_T \partial_x\left(\tilde{i}_h v - v\right)(x - x_0)\partial_{xx}\tilde{v}_h \, dx\, dy$$

$$= \partial_x \tilde{v}_h(x_0,y_0)\left(\int_{l_1} - \int_{l_2}\right)\left(\tilde{i}_h v - v\right)dy$$

$$+ \partial_{xx}\tilde{v}_h\left(\int_{l_1} - \int_{l_2}\right)(\tilde{i}_h v - v)(x - x_0)\, dy$$

$$- \partial_{xx}\tilde{v}_h \iint_T (\tilde{i}_h v - v)\, dx\, dy.$$

The first and the third term disappears and then

$$\iint_T \partial_x\left(\tilde{i}_h v - v\right)\partial_x \tilde{v}_h \, dx\, dy = \partial_{xx}\tilde{v}_h \int_{l_1} (x - x_0)\left(\tilde{i}_h v - v\right)dy.$$

By analogy,

$$\iint_T \partial_y\left(\tilde{i}_h v - v\right)\partial_y \tilde{v}_h \, dx\, dy = \partial_{yy}\tilde{v}_h \int_{l_1} (y - y_0)\left(\tilde{i}_h v - v\right)dx.$$

Having in mind that $\partial_{xx}\tilde{v}_h = \partial_{yy}\tilde{v}_h$ , we get

$$\iint_T \nabla\left(\tilde{i}_h v - v\right) \cdot \nabla \tilde{v}_h \, dx\, dy = 0.$$

Case 2. $Q_1^{rot}$ -element

Here, the presentation (5) is valid and besides

$$\partial_{xx}\tilde{v}_h = -\partial_{yy}\tilde{v}_h = const. \qquad (6)$$

We calculate

$$\iint_T \partial_x\left(\tilde{i}_h v - v\right)\partial_x \tilde{v}_h \, dx\, dy$$

$$= \partial_x \tilde{v}_h(x_0,y_0)\left(\int_{l_2} - \int_{l_4}\right)\left(\tilde{i}_h v - v\right)dy$$

$$+ \partial_{xx}\tilde{v}_h\left(\int_{l_2} - \int_{l_4}\right)(\tilde{i}_h v - v)(x - x_0)\, dy$$

$$- \partial_{xx}\tilde{v}_h \iint_T (\tilde{i}_h v - v)\, dx\, dy.$$

Thus,

$$\iint_T \partial_x\left(\tilde{i}_h v - v\right)\partial_x \tilde{v}_h \, dx\, dy = -\iint_T \left(\tilde{i}_h v - v\right)\partial_{xx}\tilde{v}_h \, dx\, dy. \qquad (7)$$

Using the same arguments, it follows:

$$\iint_T \partial_y\left(\tilde{i}_h v - v\right)\partial_y \tilde{v}_h \, dx\, dy = -\iint_T \left(\tilde{i}_h v - v\right)\partial_{yy}\tilde{v}_h \, dx\, dy. \qquad (8)$$

Finally, from (6) we obtain

$$\iint_T \nabla\left(\tilde{i}_h v - v\right) \cdot \nabla \tilde{v}_h \, dx\, dy = 0.$$

Case 3. $EQ_1^{rot}$ -element

The relation (5) is also fulfilled for any $\tilde{v}_h \in \tilde{V}_h$ as well as (7) and (8). Then

$$\iint_T \nabla\left(\tilde{i}_h v - v\right) \cdot \nabla \tilde{v}_h \, dx\, dy = -\iint_T \left(\tilde{i}_h v - v\right)\Delta \tilde{v}_h \, dx\, dy.$$

But $\Delta \tilde{v}_h = const$ and the last integral is equal to zero because of the condition

$$\iint_T \left(\tilde{i}_h v - v\right)dx\, dy = 0.$$

For all three cases we make an affine transformation from $T$ to any element $K \in \tau_h$ and summarizing over all $K$, we get the equality (4).

The Morley rectangular finite element case is considered in [10] and in this case (4) is proved therein.

## 3. MAIN RESULT

In this section we present a new simple algorithm for obtaining two-sided bounds of any exact eigenvalue $\lambda$ .

Let $(\lambda_h, u_h)$ be the eigenpair approximation obtained by (3). We consider the nonconforming interpolation of the function $u_h$ , i.e. $\tilde{i}_h u_h$ and define the number

$$\tilde{\Lambda}_h = \frac{a_h(\tilde{i}_h u_h, \tilde{i}_h u_h)}{(\tilde{i}_h u_h, \tilde{i}_h u_h)}, \qquad (9)$$

where $\|u_h\|_{0,\Omega} = 1$ .

**Theorem 1.** *Let $(\lambda_h, u_h)$ be an approximate eigenpair of the exact one $(\lambda, u)$, $u \in H^{m+2}(\Omega) \cap V$ and*

$\|u\|_{0,\Omega} = \|u_h\|_{0,\Omega} = 1$.

*Suppose that the conforming finite element space $V_h$ consists of piecewise polynomial functions of degree $n \ge m+1$, $m = 1,2$ and the corresponding space $\widetilde{V}_h$ contains nonconforming elements considered in Lemma 1. Then the number $\widetilde{\Lambda}_h$ determined by (9) ensures a lower bound of the exact eigenvalue, i.e.*

$$\widetilde{\Lambda}_h \le \lambda \le \lambda_h. \tag{10}$$

*Proof.* The upper bound in (10) is obvious because the FEM in (3) is conforming (see e.g. [4]).

Let us introduce the norms:

$\| v \|_a^2 = a(v,v),\ v \in V$ and $\| \widetilde{v}_h \|_{a_h}^2 = a(\widetilde{v}_h, \widetilde{v}_h),\ \widetilde{v}_h \in \widetilde{V}_h$.

First, we suppose that $(\widetilde{\lambda}_h, \widetilde{u}_h)$ is an approximate eigenpair corresponding to $(\lambda, u)$ and obtained by nonconforming FEM using elements described in (A) or (B), respectively. In these cases, $\widetilde{\lambda}_h$ approximates $\lambda$ from below (see [1,3,10]).

We calculate:

$$\begin{aligned}
&a_h(\widetilde{i}_h u_h - \widetilde{u}_h, \widetilde{i}_h u_h - \widetilde{u}_h) - \widetilde{\lambda}_h(\widetilde{i}_h u_h - \widetilde{u}_h, \widetilde{i}_h u_h - \widetilde{u}_h) \\
&= a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h) - 2a_h(\widetilde{i}_h u_h, \widetilde{u}_h) + a_h(\widetilde{u}_h, \widetilde{u}_h) \\
&- \widetilde{\lambda}_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h) + 2\widetilde{\lambda}_h(\widetilde{i}_h u_h, \widetilde{u}_h) - \widetilde{\lambda}_h(\widetilde{u}_h, \widetilde{u}_h) \\
&= \left(\widetilde{\Lambda}_h - \widetilde{\lambda}_h\right) \left\| \widetilde{i}_h u_h \right\|_{0,\Omega}^2.
\end{aligned}$$

So that, we obtain:

$$\widetilde{\Lambda}_h - \widetilde{\lambda}_h = \frac{\left\| \widetilde{i}_h u_h - \widetilde{u}_h \right\|_{a_h}^2 - \widetilde{\lambda}_h \left\| \widetilde{i}_h u_h - \widetilde{u}_h \right\|_{0,\Omega}^2}{\left\| \widetilde{i}_h u_h \right\|_{0,\Omega}^2}.$$

This equality shows that $\widetilde{\Lambda}_h > \widetilde{\lambda}_h$ asymptotically, because the function $\widetilde{i}_h u_h - \widetilde{u}_h$ is a piecewise polynomial belonging to $\widetilde{V}_h$.

Consequently, we have to estimate

$$\left\| \widetilde{i}_h u_h - \widetilde{u}_h \right\|_{a_h} \le \left\| \widetilde{i}_h u_h - \widetilde{i}_h u \right\|_{a_h} + \left\| \widetilde{i}_h u - \widetilde{u}_h \right\|_{a_h}.$$

The interpolation operator $\widetilde{i}_h : V \to \widetilde{V}_h$ has a finite range, i.e. $dim\, range(\widetilde{i}_h) < \infty$. Therefore it is compact.

Thus

$$\left\| \widetilde{i}_h \right\| = sup_{v \in V} \frac{\left\| \widetilde{i}_h v \right\|_{m,\Omega}}{\| v \|_{m,\Omega}} \le C = const.$$

It follows that

$$\left\| \widetilde{i}_h u_h - \widetilde{u}_h \right\|_{a_h}^2 \le C\left( \| u_h - u \|_a^2 + \left\| \widetilde{i}_h u - \widetilde{u}_h \right\|_{a_h}^2 \right). \tag{11}$$

The space $V_h$ contains piecewise polynomial functions of degree at least $n = m + 1$ and

$$\| u - u_h \|_a^2 = O\left(h^{2m+2}\right),\ m = 1;2.$$

On the other hand $\widetilde{i}_h$ coincides with the elliptic projection operator on $\widetilde{V}_h$ denoted by $\widetilde{R}_h$ and verifying

$$a_h(\widetilde{R}_h v, \widetilde{v}_h) = (v, \widetilde{v}_h),\ \forall v \in V,\ \widetilde{v}_h \in \widetilde{V}_h.$$

Indeed, from (4) ($\alpha = const > 0$:

$$\begin{aligned}
&\alpha \left\| \widetilde{i}_h v - \widetilde{R}_h v \right\|_{m,h}^2 \le a_h(\widetilde{i}_h v - \widetilde{R}_h v, \widetilde{i}_h v - \widetilde{R}_h v) \\
&= a_h(\widetilde{i}_h v - v, \widetilde{i}_h v - \widetilde{R}_h v) = 0,
\end{aligned}$$

where $\| \cdot \|_{m,h}$ is the mesh-dependent *m*-th norm.

For the nonconforming finite elements (A), (B) under consideration we have (see [1,10,12,13])

$$\lambda - \widetilde{\lambda}_h = O\left( \| u - \widetilde{u}_h \|_a^2 \right) \le C h^2 \| u \|_{m+1}^2. \tag{12}$$

The elliptic operator $\widetilde{R}_h$ fulfils a superclose property with the corresponding finite element eigenvector [11] (see also [13]):

$$\left\| \widetilde{i}_h u - u_h \right\|_{a_h} = \left\| \widetilde{R}_h u - u_h \right\|_{a_h} \le C h^2 \| u \|_{m+2,\Omega}^2. \tag{13}$$

Thus from (11), using (13), we obtain

$$\left\| \widetilde{i}_h u_h - \widetilde{u}_h \right\|_{a_h}^2 \le C h^4.$$

Consequently

$$\widetilde{\Lambda}_h - \widetilde{\lambda}_h \le C h^4.$$

This inequality and (12) give for *h* sufficiently small:

$$\widetilde{\Lambda}_h - \lambda \le 0,$$

which proves (10).

□

So, we can propose the following

**Algorithm**

1. Solve the discrete eigenvalue problem (3) by means of conforming FEM and find an eigenpair $(\lambda_h, u_h)$;
2. Construct a nonconforming interpolant of $u_h$ using convenable basis discussed in Lemma 1. It is preferable to use integral type degrees of freedom in $\widetilde{V}_h$ such that they take part in $V_h$, too;
3. Calculate $\widetilde{\Lambda}_{h,j}$ according to (9). Then

$$\lambda \in [\widetilde{\Lambda}_h, \lambda_h].$$

## 4. NUMERICAL RESULTS

To illustrate the theoretical results we report in this example on related second-order eigenvalue problem:

$$-\Delta u=\lambda u \text{ in } \Omega,$$
$$u=0 \quad \text{on } \partial\Omega,$$

where $\Omega=(0,\pi)\times(0,\pi)$.

For this problem the exact eigenvalues are

$$\lambda_j=s_1^2+s_2^2,\ s_{1,2}=1,2,3,...$$

In Table 1 the results from numerical experiments for the first four eigenvalues are given. Their exact values are equal to *2, 5, 5, 8*, respectively. The domain is uniformly divided into $2n^2$ isosceles triangles and thus the mesh parameter is *1/n, n=4,8,12,16*. The numerical results $\lambda_{h,j}$ obtained by means of *6*-node triangular elements and the resulting numbers $\widetilde{\Lambda}_{h,j}$ after nonconforming *C-R*-interpolation of the conforming FE solution are compared with those obtained solving the eigenvalue problem with nonconforming *C-R* elements $\widetilde{\lambda}_{h,j}, j=1,2,3,4$.

Table 1 illustrates that both eigenvalue sequences $\widetilde{\Lambda}_{h,j}$ and $\widetilde{\lambda}_{h,j}$ are increasing and the first sequence is greater than the second one, which verifies the theoretical results.

**Table 1.** *Eigenvalue approximations* $\lambda_{h,j}$ *computed by means of 6-node quadratic triangular conforming FEs, values* $\widetilde{\Lambda}_{h,j}$ *obtained as a result of nonconforming C-R interpolation of the conforming approximate FE solution and eigenvalue approximations* $\widetilde{\lambda}_{h,j}$ *by means of nonconforming C-R FEs*

| *n* | | *j = 1* | *j = 2* | *j = 3* | *j = 4* |
|---|---|---|---|---|---|
| 4 | $\lambda_{h,j}$ | 2.0066781 | 5.0541368 | 5.1049165 | 8.3228101 |
| | $\widetilde{\Lambda}_{h,j}$ | 1.9978343 | 4.9692132 | 4.9659029 | 7.9174807 |
| | $\widetilde{\lambda}_{h,j}$ | 1.9654755 | 4.5460329 | 4.5460365 | 7.4309499 |
| 8 | $\lambda_{h,j}$ | 2.0004496 | 5.0040458 | 5.0074545 | 8.0266116 |
| | $\widetilde{\Lambda}_{h,j}$ | 1.9998512 | 4.9979470 | 4.9972997 | 7.9916007 |
| | $\widetilde{\lambda}_{h,j}$ | 1.9914177 | 4.8881333 | 4.8881346 | 7.8689405 |
| 12 | $\lambda_{h,j}$ | 2.0000902 | 5.0008288 | 5.0015185 | 8.0055801 |
| | $\widetilde{\Lambda}_{h,j}$ | 1.9997004 | 4.9995861 | 4.9994428 | 7.9982068 |
| | $\widetilde{\lambda}_{h,j}$ | 1.9961894 | 4.9504042 | 4.9504053 | 7.9446002 |
| 16 | $\lambda_{h,j}$ | 2.0000287 | 5.0002657 | 5.0004860 | 8.0018049 |
| | $\widetilde{\Lambda}_{h,j}$ | 1.9999996 | 4.9999990 | 4.9999989 | 7.9997821 |
| | $\widetilde{\lambda}_{h,j}$ | 1.9978572 | 4.9721260 | 4.9721271 | 7.9710044 |

Regardless of the fact, that the second and the third eigenvalues are equal, the proposed theoretical results are valid for both of them.

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